Indian Statistical Institute, Bangalore

B.Math (Hons.) II Year, Second Semester

Semestral Examination, Back Paper

Optimization

Time: 3 hours May 00, 2011 Instructor: Pl.Muthuramalingam

Maximum marks: 50

1. Let $R_n^{++} = \{\mathbf{p} = (p_1, p_2, \cdots p_n)$ with $p_i > 0$ for each $i.\}$ Let $\triangle_n = \{(x_1, x_2, \cdots x_n) : x_i \geq 0$ for each i and $\sum x_i = 1\}$. Define for $\mathbf{p}\varepsilon R_n^{++}$ $S_{\mathbf{p}} : \triangle_n \longrightarrow \triangle_n$ by $S\mathbf{p}(x_1, x_2, \cdots, x_n) = \frac{(p_1x_1, p_2x_2, \cdots p_nx_n)}{\sum p_jx_j}$

   a) Let $\mathbf{p}, \mathbf{q}\varepsilon R_n^{++}$. Let $\mathbf{r} = (r_1, r_2, \cdots r_n)$ with $r_j = p_jq_j$. Find a relation between $S_{\mathbf{p}} \circ S_{\mathbf{q}}$ and $S_{\mathbf{r}}$. [3]

   b) Show that $S_{\mathbf{p}}$ is $1-1$, onto for each $\mathbf{p}$ in R_n^{++}. [1]

   c) Show that $S\mathbf{p}$ maps any straight line in $\triangle_n$ to a straight line. [3]

2. State and prove weak duality theorem for the standard LP form

   TABLE

Primal	Dual
row $i \sum_j a_{ij}x_j = b_i$	y_i real
row$p \sum_j a_{pj}x_j \geq b_p$	$y_p \geq 0$
variable j x_j real	$\sum_i y_ia_{ij} = c_j$
var q $x_q \geq 0$	$\sum_i y_ia_{iq} \leq c_q$
min $\sum c_jx_j$	max $\sum y_ib_i$

   form [5]

3. Define $f, g : R^2 \longrightarrow R, L : R^2 \times R \longrightarrow R$ by

   $f(x, y), = 2x^3 - 3x^2$

   $g(x, y) = (3 - x)^3 - y^2$

   $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$.

   Let D = {(x,y) : g(x,y) = 0}

   a) Show that $\max_D f$ is attained at $(3, 0)$. [3]

   b) If (x, y, λ) is a critical point for L ie $\partial_xL, \partial_yL, \partial_\lambda L = 0$ at (x, y, λ), then show that $(x, y, \lambda) \in \{(0, \pm\sqrt{27}, 0), (1, \pm\sqrt{8}, 0)\}$. [2]

   c) Find the rank of g' at $(3, 0)$. [1]

4. State Lagrange theorem. [3]

5. State Kuhn-Tucker theorem for C^1 functions $f; R^n \longrightarrow R, h_1, h_2, h_l : R^n \to R$. [5]

6. Let
$$P = \left\{ \begin{array}{ll} (x_1, x_2) \in R^2 : & x_1 + x_2 \leq 40, \\ & 2x_1 + x_2 \leq 60 \\ & x_1 \leq 20 \\ & x_1, x_2 \geq 0 \end{array} \right\}.$$

a) Show that
$$P = \left\{ \begin{array}{ll} (x_1, x_2) \in R^2 : & x_1 + x_2 \leq 40 \\ & x_1 \leq 20 \\ & x_1 x_2 \leq 0 \end{array} \right\}.$$
[1]

b) Draw the figure if P and find its extreme points. [1]

c) Convert P into standard equality form. [1]

d) For (c) find all the bases. [2]

e) Find all basic solutions. [6]

f) Find all basic feasible solutions. [1]

g) Find all basic feasible non degenerate solutions. [1]

h) Find all degenerate basic feasible solutions. [1]

i) In (b) move from one vertex to next vertex in anticlock wise direction. [1]

j) For each extreme point in (b), find corresponding point in (f). [1]

7. a) Let $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{z} \geq \mathbf{0}$ in R^n. Show that $\mathbf{x} \cdot \mathbf{z} \geq \mathbf{y} \cdot \mathbf{z}$. [1]

b) Let $S1, S2$ be statements given below for $A : R^n_{col} \longrightarrow R^m_{col}$ linear, on to, $\mathbf{b} \in R^m_{col}$, $\quad \mathbf{v} \in R^n_{col}$ and $\mathbf{y} \in R^n_{col}$.

S1. There exists $\mathbf{v}$ in R^n_{col} such that $A\mathbf{v} \leq \mathbf{b}$

S2. There exists $\mathbf{y}$ in R^m_{col} satisfying $\mathbf{y} \geq \mathbf{0}$, $\quad A^t\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^t \cdot \mathbf{y} < 0$. Show that $S1$ and $S2$ cannot hold simultaneously. [2]

8. a) Let $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_k \in R^n$ and $S = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k : \lambda_i \geq 0, \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1\}$. Let $f : R^n \longrightarrow R$ be any linear map. Show that $\max_S f = \max_i f(\mathbf{v}_i)$. [2]

b) Let P be a polygonal region in R^2 given by

$$2x_1 + x_2 \geq 4$$

$$x_1 - x_2 \geq -4$$

$$-3x_1 + x_2 \geq -15$$

$$-x_1 \geq -7$$

$$x_1 \geq 0, x_2 \geq 0.$$

Let $c_1, c_2 \in R$. Define $g : R^2 \longrightarrow R$ by $g(x_1, x_2) = c_1x_1 + c_2x_2$. Determine $\max\limits_{P} g$ and $\min\limits_{P} g$ interms of c_1, c_2. [3]